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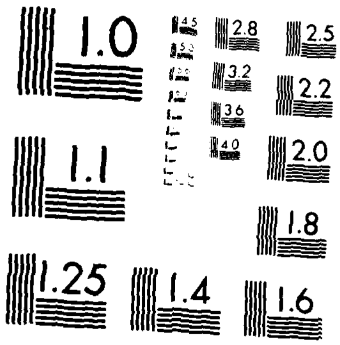

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Technical Note BN-1004

THE APPROXIMATION THEORY FOR THE P-VERSION OF THE FINITE ELEMENT METHOD, I

by

Milo R. Dorr

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In its standard mathematical formulation, the finite element method is a particular kind of Ritz-Galerkin procedure in which the approximating finite-dimensional subspaces are composed of piecewise polynomials defined on a partition of the given domain into convex subdomains. Since the convergence of such methods is obtained by increasing the dimension of these subspaces in some manner, one observes that there are basically two ways this can be done. The first way is the traditional approach obtained by fixing the degree p of the piecewise polynomials at some value $(p = 1,2,3)$ and decreasing the

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THE APPROXIMATION THEORY FOR THE P-VERSION

OF THE FINITE ELEMENT METHOD, I

Milo R. Dorr

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1. Introduction

In its standard mathematical formulation, the finite element method is a particular kind of Ritz-Galerkin procedure in which the approximating finite-dimensional subspaces are composed of piecewise polynomials defined on a partition of the given domain into convex subdomains. Since the convergence of such methods is obtained by increasing the dimension of these subspaces in some manner, one observes that there are basically two ways this can be done. The first way is the traditional approach obtained by fixing the degree p of the piecewise polynomials at some value $(p = 1,2,3)$ and decreasing the mesh size h in order to achieve convergence; this is known as the h-version of the finite element method. The second way, referred to as the p-version of the finite element method, is to fix the mesh and increase the degree p in order to reduce the approximation error. Clearly, a combination of the two is also possible. While the h-version has been extensively investigated in the mathematical literature and has been widely used in engineering applications for many years, the development of the p-version has taken place only recently. Due largely to the investigations performed at the Center for Computational Mechanics at Washington University in St. Louis, it is now recognized (e.g. [4], [12], [16]) that for many problems of engineering and scientific interest, the p-version offers a number of advantages over the h-version both in the quality of approximation and in the cost of computation.

In the mathematical analysis of either the h or p

versions, it is well-known that, if a coercivity or "inf-sup" condition can be established for the given problem, then the task of obtaining error estimates reduces to a purely approximation-theoretic question (see e.g. [2], [8]). For the h-version, this approximation theory is very well-developed. On the other hand, the approximation issues arising in the p-version require different techniques and have not been as thoroughly investigated. In [4], some direct energy norm estimates are obtained which show that the rate of convergence for the p-version can be no worse than that of the h-version with a quasi-uniform sequence of mesh refinements. A combination of the h and p versions is considered in [3] where it is demonstrated that particular couplings of refined meshes and increasing polynomial degree distributions yield arbitrarily high rates of convergence in the energy norm with respect to the number of degrees of freedom. However, both of these analyses fail to predict the improved (by a factor of 2) rate of convergence which is observed in applications of the p-version to various problems of two-dimensional linear elasticity where singularities are commonly present in the solutions. By applying a separate analysis to the known singularities of such problems, the doubled rate of convergence is also proven in [4], although the techniques are rather specialized and do not seem to readily extend to the three-dimensional case or to problems in which the solution is singular at more than just a finite set of isolated points. Finally, a number of somewhat related approximation results have been obtained in the analysis of an

alternative to finite elements and finite differences known as the spectral method (see e.g. [7] and the references contained therein).

The purpose of this two-part paper is to attempt to unify the p-version approximation theory by establishing a framework from which many of the above and other results may be derived. The present article addresses the issue of piecewise polynomial approximation on triangulated domains of $\mathbb{R}^n$. The application of results obtained here to problems of two- and three-dimensional linear elasticity, including some numerical computations, will be given in the second article.

A key idea in the following development is the introduction of certain weighted Sobolev spaces, which are identified in section 2 as the domains of powers of the Legendre differential operator. Their connection with polynomial approximation is obtained by exploiting the fact that the eigenfunctions of the Legendre operator are themselves polynomials. This one-dimensional result is then readily extended via a tensor product construction to obtain approximation results on any triangulated domain in $\mathbb{R}^n$, provided that no compatibility conditions are required across the common boundaries of adjacent simplices. In many applications, however, one must use piecewise polynomials which possess a certain number of continuous derivatives across the common boundaries of adjacent simplices. Moreover, the approximating functions will often be required to satisfy a set of boundary conditions associated with the underlying problem. In the finite element literature, such piecewise

polynomials are said to be conforming. In sections 3, 4, and 5, it is shown that, up to an arbitrarily small $\epsilon > 0$, one can obtain conforming piecewise polynomials yielding the same degree of approximation as the non-conforming functions of section 2 provided that the function being approximated satisfies the same boundary conditions and compatibility conditions across the common boundaries of adjacent simplices. Moreover, for the special case of approximation in L_2, the latter condition is also necessary. A precise statement of these results is given in section 3 (Theorems 3.1, 3.2, and 3.3) with the proofs for the one- and two-dimensional cases presented in sections 4 and 5, respectively.

2. Approximation by non-conforming piecewise polynomials on triangulated domains

In this section, piecewise polynomial approximation is considered on triangulated domains of $\mathbb{R}^n$. The approximating functions are not required to satisfy any compatibility conditions across the common boundaries of adjacent simplices of the triangulation, and hence the central issue is that of polynomial approximation on an individual simplex.

Let I denote the interval $-1 < t < 1$ and let $C^\infty(\bar{I})$ be the set of all infinitely differentiable functions on $\bar{I}$. Regarding the Legendre differential operator

$$L = -\frac{d}{dt}[(1-t^2)\frac{d}{dt}]$$

as a symmetric, unbounded operator in $L_2(I)$ with domain of definition $C^\infty(\bar{I})$, it is shown in [15, Theorem 7.4.1] that the closure $\bar{L}$ of L is self-adjoint. In fact, since L is non-negative, $\bar{L}$ coincides with the Friedrichs extension of L as constructed, for example, in [17]. It is well-known that L (and hence $\bar{L}$) possesses the eigenvalues

$$\ell_m = m(m+1), \qquad m = 0,1,\ldots$$

and that the corresponding eigenfunctions are the Legendre polynomials P_m. Assuming that the P_m have been normalized so that $\|P_m\|_{L_2(I)} = 1$ for all m, the system $\{P_m\}$ forms an orthonormal basis for $L_2(I)$.

Given any real $s \geq 0$, define

$$Z^s(I) = \{u: \|u\|_{Z^s(I)} < \infty\}$$

where, if $s = k$ an integer,

$$\|u\|_{Z^s(I)} = \left(\int_I |u|^2 dt + \int_I \left|\frac{d^k u}{dt^k}\right|^2 (1-t^2)^k \, dt\right)^{1/2},$$

and if $s = k + \beta$ with k an integer and $0 < \beta < 1$,

$$\|u\|_{Z^s(I)} = \left(\|u\|^2_{Z^k(I)} + \int_{I\times I} \frac{\left|(1-t^2)^{s/2} \frac{d^k u}{dt^k} - (1-\tau^2)^{s/2} \frac{d^k u}{d\tau^k}\right|^2}{|t-\tau|^{1+2\beta}} \, dt d\tau\right)^{1/2}.$$

Denoting by $D(\bar{L}^{s/2})$ the domain of definition of $\bar{L}^{s/2}$ in $L_2(I)$ *for each* $s \geq 0$, *one has the following result.*

Lemma 2.1. (i) $C^\infty(\bar{I})$ is dense in $Z^s(I)$ for all $s \geq 0$,

(ii) $Z^s(I) = D(\bar{L}^{s/2})$ for all $s \geq 0$ such that $s \neq \frac{1}{2}$ + an integer, and

(iii) if $s_1, s_2 \geq 0$ are such that $s_i \neq \frac{1}{2}$ + an integer, $i = 1,2$, and if $0 < \theta < 1$ is such that $s = (1-\theta)s_1 + \theta s_2 \neq \frac{1}{2}$ + an integer, then[†]

$$(Z^{s_1}(I), Z^{s_2}(I))_{\theta,2} = Z^s(I).$$

[†] *For* $0 < \theta < 1$ *and* $1 \leq q \leq \infty$, $(\cdot,\cdot)_{\theta,q}$ *denotes real interpolation via the K-method (see e.g. [6]).*

Pf: Part (i) is proved in [13] for integer s and in [14] for non-integer s. Part (ii) is contained in [15, Theorem 7.7.1]. By [15, Theorem 1.18.10], if T is any non-negative, self-adjoint operator, then for all $s_1, s_2 \geq 0$ and $0 < \theta < 1$ it holds that

$$(D(T^{s_1}), D(T^{s_2}))_{\theta,2} = D(T^{(1-\theta)s_1+\theta s_2}).$$

Applying this to $T = \bar{L}^{1/2}$, (iii) follows from (ii).

Remark. One observes that, except for the values $s = \frac{1}{2}$ + an integer, the spaces $Z^s(I)$ form a Hilbert scale. Regarding the values $s = \frac{1}{2}$ + an integer, Triebel [15] modifies the spaces $Z^s(I)$ to identify $D(\bar{L}^{s/2})$ in these special cases. More specifically, it is shown that for $s = \frac{1}{2} + k$, $D(\bar{L}^{s/2})$ is the completion of $C^\infty(\bar{I})$ in the norm

$$\|u\|_{\tilde{Z}^s(I)} = \left(\|u\|^2_{Z^s(I)} + \int_I \left|\frac{d^k u}{dt^k}\right|^2 (1-t^2)^{s-1}\, dt\right)^{1/2}.$$

This anomaly is similar to that encountered in attempting to identify the Sobolev space $\overset{\circ}{H}{}^{\frac{1}{2}+k}$ as the domain of definition of a power of the negative Laplacian with homogeneous boundary data.

The following technical lemma will be of use in obtaining subsequent results,

Lemma 2.2. For $\alpha \neq 1$,

$$\int_I |u(t)-a|^2 t^{\alpha-2}\, dt \leq C(\alpha) \int_I \left|\frac{du}{dt}\right|^2 t^\alpha\, dt \tag{2.1}$$

where $a = u(0)$ if $\alpha < 1$, $a = u(1)$ if $\alpha > 1$.

Pf: Suppose that $\alpha < 1$ and let $w(s) = u(s^{\frac{1}{1-\alpha}}) - u(0)$. Then since $w(0) = 0$, one has by [9, Theorem 254] that

$$\int_0^1 |w(s)|^2 s^{-2} ds \le C\left(\int_0^1 |w'(s)|^2 ds + |w(1)|^2\right).$$

Since $w(1) = \int_0^1 w'(s)ds$, it follows that

$$\int_0^1 |w(s)|^2 s^{-2} ds \le C \int_0^1 |w'(s)|^2 ds$$

and (2.1) follows by making the change of variable $s = t^{1-\alpha}$.

Now suppose that $\alpha > 1$ and let

$$w(s) = \begin{cases} u(s^{\frac{1}{1-\alpha}}) - u(1) & \text{if } 1 \le s < \infty \\ 0 & \text{if } 0 \le s \le 1 . \end{cases}$$

Then, since $w(0) = 0$, one obtains by [9, Theorem 253] that

$$\int_0^\infty |w(s)|^2 s^{-2} ds \le C \int_0^\infty |w'(s)|^2 ds$$

which yields (2.1) after again making the change of variable $s = t^{1-\alpha}$.

For any non-negative real number s, let $H^s(I)$ denote the usual Sobolev space of order s on I which, for integer values of s, is defined as the completion of $C^\infty(\bar{I})$ in the norm

$$\|u\|_{H^s(I)} = \left(\int_I |u|^2 dt + \int_I \left| \frac{d^s u}{dt^s} \right|^2 dt \right)^{1/2}.$$

and for non-integer s is defined by real interpolation between the integer-ordered spaces. The next lemma gives the relationship between the unweighted spaces $H^s(I)$ and the weighted spaces $Z^s(I)$.

<u>Lemma 2.3</u>. If s is any non-negative real number such that $s \neq \frac{1}{2}$ + an integer, then $Z^s(I)$ is continuously imbedded in $H^{s/2}(I)$.

<u>Pf</u>: Consider first the case in which $s = 2k$, k a positive integer. It suffices to prove that for any $u \in C^\infty(\bar{I})$,

$$\|u\|_{H^k(I)} \leq C\|u\|_{Z^{2k}(I)} \tag{2.2}$$

with C independent of u. Hence, let $X_1 \in C^\infty(\bar{I})$ be such that

$$X_1(t) = \begin{cases} 1 & -1 \leq t \leq \frac{1}{3} \\ 0 & \frac{1}{3} \leq t \leq 1 \end{cases}$$

and let $u_1 = uX_1$. By Leibniz' rule together with repeated application of Lemma 2.2 (with the appropriate scaling) one obtains that

$$\int_I \left|\frac{d^k u_1}{dt^k}\right|^2 dt \le \int_I \left|\frac{d^{2k} u_1}{dt^{2k}}\right|^2 (1+t)^{2k}\, dt$$

$$\le C\left(\sum_{\ell=0}^{2k-1} \int_{-1/3}^{1/3} \left|\frac{d^\ell u}{dt^\ell}\right|^2 dt + \int_{-1}^{1/3} \left|\frac{d^{2k} u}{dt^{2k}}\right|^2 (1+t)^{2k}\, dt\right).$$

Applying a standard interpolation inequality (see e.g. [10]), it follows that

$$\int_I \left|\frac{d^k u_1}{dt^k}\right|^2 dt \le C\left\{\sum_{\ell=0}^{2k-1}\left(\int_{-1/3}^{1/3} |u|^2 dt + \int_{-1/3}^{1/3} \left|\frac{d^{2k} u}{dt^{2k}}\right|^2 dt\right) + \int_{-1}^{1/3} \left|\frac{d^{2k} u}{dt^{2k}}\right|^2 (1+t)^{2k}\, dt\right\}$$

$$\le C\|u\|^2_{Z^{2k}(I)}$$

and hence,

$$\|u\chi_1\|_{H^k(I)} \le C\|u\|_{Z^{2k}(I)}. \tag{2.3}$$

Letting $\chi_2 = 1 - \chi_1$, one similarly shows that

$$\|u\chi_2\|_{H^k(I)} \le C\|u\|_{Z^{2k}(I)}$$

which, together with (2.3) implies (2.2). Since $H^0(I) = Z^0(I) = L_2(I)$, the result for all s satisfying the hypothesis of the lemma follows via interpolation (see e.g. [6]).

For each positive integer n, consider the cube $I^n = \{x = (x_1, x_2, \ldots, x_n) : -1 < x_i < 1,\ 1 \le i \le n\}$ in $\mathbb{R}^n$. If

denotes the identity operator in $L_2(I)$ and s is a non-negative real number, define a differential operator Λ_s in $L_2(I^n)$ by

$$\Lambda_s = \bar{L}^{s/2} \otimes E \otimes \cdots \otimes E + E \otimes \bar{L}^{s/2} \otimes \cdots \otimes E \tag{2.4}$$
$$+ \cdots + E \otimes E \otimes \cdots \otimes \bar{L}^{s/2}$$

where each of the tensor products (defined e.g. as in [11]) in the n terms of the right-hand side of (2.4) contains n factors. The following result is one part of [11, Corollary to Theorem VIII.33] applied to the closed, non-negative, self-adjoint operators $\bar{L}^{s/2}$.

Lemma 2.4. (i) Λ_s is a non-negative and self-adjoint operator in $L_2(I^n) = \bigotimes_{i=1}^{n} L_2(I)$ with domain of definition

$$D(\Lambda_s) = D(\bar{L}^{s/2}) \otimes L_2(I) \otimes \cdots \otimes L_2(I) \cap L_2(I) \otimes D(\bar{L}^{s/2})$$
$$\otimes \cdots \otimes L_2(I) \cap \cdots \cap L_2(I) \otimes L_2(I) \otimes \cdots \otimes D(\bar{L}^{s/2}).$$

(ii) Λ_s possesses the eigenvalues

$$\lambda_{\underline{m}}^{(s)} = \sum_{i=1}^{n} \ell_{m_i}^{s/2} = \sum_{i=1}^{n} [m_i(m_i+1)]^{s/2}, \qquad \underline{m} = (m_1,\ldots,m_n).$$

(iii) The eigenfunctions of Λ_s are

$$\Phi_{\underline{m}}(x) = \prod_{i=1}^{n} P_{m_i}(x_i), \qquad \underline{m} = (m_1,\ldots,m_n),$$

and the system $\{\Phi_{\underline{m}}\}$ is an orthonormal basis for $L_2(I^n)$.

For each positive real number s such that $s \neq \frac{1}{2}$ + an integer, define

$$Z^s(I^n) = Z^s(I) \otimes L_2(I) \otimes \cdots \otimes L_2(I) \cap L_2(I) \otimes Z^s(I) \otimes \cdots \otimes L_2(I) \cap \cdots \cap L_2(I) \otimes L_2(I) \otimes \cdots \otimes Z^s(I).$$

The following is a special case of [11, Theorem II.10(b)].

<u>Lemma 2.5</u>. Let H be a separable Hilbert space. If $L_2(I;H)$ denotes the Hilbert space of measurable functions u on I with values in H such that

$$\|u\|_{L_2(I;H)} = \left(\int_I \|u(t)\|_H^2 \, dt\right)^{1/2} < \infty,$$

then there exists a unique isomorphism from $L_2(I) \otimes H$ onto $L_2(I;H)$ such that $u(x) \otimes \varphi \to u(x)\varphi$.

As a consequence of Lemma 2.5 and Fubini's theorem, it follows that the space $Z^s(I^n)$ may be equivalently defined as

$$Z^s(I^n) = \{u: \|u\|_{Z^s(I^n)} < \infty\}$$

where, if $s = k$ an integer, then

$$\|u\|_{Z^k(I^n)} = \left(\int_{I^n} |u|^2 dx + \sum_{i=1}^n \int_{I^n} \left|\frac{\partial^k u}{\partial x_i^k}\right|^2 (1-x_i^2)^k \, dx\right)^{1/2},$$

and if $s = k + \beta$ with k an integer and $0 < \beta < 1$, then

$$\|u\|_{Z^s(I^n)} = \Bigg(\|u\|^2_{Z^k(I^n)} + \sum_{i=1}^{n} \int_{I^{n-1}}$$

$$\cdot \Bigg(\int_{I\times I} \Big| (1-t^2)^{s/2} \frac{\partial^k u}{\partial x_i^k} (x_1,\ldots,x_{i-1},t,x_{i+1},\ldots,x_n)$$

$$- (1-\tau^2)^{s/2} \frac{\partial^k u}{\partial x_i^k} (x_1,\ldots,x_{i-1},\tau,x_{i+1},\ldots,x_n)\Big|^2$$

$$\cdot |t-\tau|^{-1-2\beta}\, dt d\tau \Bigg) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n \Bigg)^{1/2} .$$

Let $C^\infty(\overline{I^n})$ denote the space of all infinitely differentiable functions on $\overline{I^n}$. The following generalizes Lemma 2.1.

<u>Theorem 2.1</u>. (i) $C^\infty(\overline{I^n})$ is dense in $Z^s(I^n)$ for all $s \geq 0$,

(ii) $Z^s(I^n) = D(\Lambda_s)$ for all $s \geq 0$ such that $s \neq \frac{1}{2}$ + an integer, and

(iii) if $s_1, s_2 \geq 0$ are such that $s_i \neq \frac{1}{2}$ + an integer, $i = 1,2$, and if $0 < \theta < 1$ is such that $s = (1-\theta)s_1 + \theta s_2 \neq \frac{1}{2}$ + an integer, then

$$(Z^{s_1}(I^n), Z^{s_2}(I^n))_{\theta,2} = Z^s(I^n).$$

<u>Pf</u>: Parts (i) and (ii) follow from the definition of $Z^s(I^n)$ and from Lemmas 2.1 and 2.4. In order to prove (iii), one first observes that, for $\sigma \geq 0$,

$$D(\Lambda_\sigma) = D(\Lambda_1^\sigma)$$

(although $\Lambda_\sigma \neq \Lambda_1^\sigma$). Hence, since Λ_1 is non-negative and self-adjoint, it follows that for $s_1, s_2 \geq 0$ and $0 < \theta < 1$

$$(D(\Lambda_{s_1}), D(\Lambda_{s_2}))_{\theta,2} = (D(\Lambda_1^{s_1}), D(\Lambda_1^{s_2}))_{\theta,2}$$

$$= D(\Lambda_1^{s_1(1-\theta)+s_2\theta})$$

$$= D(\Lambda_s).$$

This together with (ii) yields (iii).

Letting $H^s(I^n)$ denote the usual Sobolev space of order s on I^n, the following is a consequence of Lemma 2.3 and the definition of the spaces $Z^s(I^n)$.

<u>Lemma 2.6</u>. If s is any non-negative real number such that $s \neq \frac{1}{2}$ + an integer, then $Z^s(I^n)$ is continuously imbedded in $H^{s/2}(I^n)$.

From the above results, one observes that if E denotes the identity in $L_2(I)$, then $(\bar{L}+E)^{-1}$ exists in $L_2(I)$ and $(\bar{L}+E)^{-1}: L_2(I) \to D(\bar{L}) = Z^2(I)$. By Lemma 2.3, it holds that $Z^2(I)$ is continuously imbedded in $H^1(I)$, which in turn is compactly imbedded in $L_2(I)$. Hence, it follows that $(\bar{L}+E)^{-1}$ is compact as well as self-adjoint in $L_2(I)$. Consequently, the spectrum of $\bar{L}$ consists only of the eigenvalues ℓ_m, and moreover, for each $u = \sum_{m=0}^{\infty} a_m P_m \in D(\bar{L})$,

$$\bar{L}u = \sum_{m=0}^{\infty} \ell_m a_m P_m .$$

Since $\bar{L}$ is self-adjoint and non-negative, one obtains that for each real $s \geq 0$, if $u = \sum_{m=0}^{\infty} a_m P_m \in D(\bar{L}^{s/2})$, then

$$\bar{L}^{s/2}u = \sum_{m=0}^{\infty} \ell_m^{s/2} a_m \varphi_m .$$

By Lemma 2.4 and Theorem 2.1, this yields that if $u = \sum_{|\underline{m}|=0}^{\infty} a_{\underline{m}}\varphi_{\underline{m}} \in L_2(I^n)$, then $u \in Z^s(I^n)$ for $s \neq \frac{1}{2}$ + an integer if and only if

$$(2.5) \qquad \left(\sum_{|\underline{m}|=0}^{\infty} a_{\underline{m}}^2 [1+(\lambda_{\underline{m}}^{(s)})^2] \right)^{1/2} < \infty$$

where, for $\underline{m} = (m_1, \ldots, m_n)$, $|\underline{m}| = \sum_{i=1}^{n} m_i$. Since $\lambda_{\underline{m}}^{(s)} = \sum_{i=1}^{n} [m_i(m_{i+1})]^{s/2}$, (2.5) is easily shown to be equivalent to

$$(2.6) \qquad \left(\sum_{|\underline{m}|=0}^{\infty} a_{\underline{m}}^2 [1 + \sum_{i=1}^{n} m_i^{2s}] \right)^{1/2} < \infty .$$

In fact, (2.6) defines an equivalent norm in $Z^s(I^n)$ for $s \neq \frac{1}{2}$ + an integer.

For each non-negative integer p, let $P_p(I^n)$ denote the space of all polynomials on I^n of degree at most p.

<u>Theorem 2.2</u>. Let s and s' be such that $s > s' \geq 0$ and $s,s' \neq \frac{1}{2}$ + an integer. If $u \in Z^s(I^n)$ then for each non-negative integer p there exists $\varphi_p \in P_p(I^n)$ such that

$$\|u-\varphi_p\|_{Z^{s'}(I^n)} \leq Cp^{-s+s'} \|u\|_{Z^s(I^n)}$$

where $C = C(s,s')$ is independent of u and p.

Pf: Let $u = \sum_{|\underline{m}|=0}^{\infty} a_{\underline{m}}\phi_{\underline{m}}$ and for each non-negative integer p let $\varphi_p = \sum_{|\underline{m}|=0}^{p} a_{\underline{m}}\phi_{\underline{m}}$. Since $\phi_{\underline{m}} \in P_p(I^n)$ for $0 \leq |\underline{m}| \leq p$, one has that $\varphi_p \in P_p(I^n)$ and

$$\|u-\varphi_p\|_{Z^{s'}(I^n)} \leq C \sum_{|\underline{m}|>p} a_{\underline{m}}^2[1 + \sum_{i=1}^{n} m_i^{2s'}]$$

$$\leq Cp^{-2(s-s')} \sum_{|\underline{m}|>p} a_{\underline{m}}^2[1 + \sum_{i=1}^{n} m_i^{2s}]$$

$$\leq Cp^{-2(s-s')}\|u\|^2_{Z^s(I^n)},$$

which completes the proof.

The following result is the inverse of Theorem 2.2, up to an arbitrarily small $\varepsilon > 0$.

Theorem 2.3. Let s and s' be such that $s > s' \geq 0$ and $s,s' \neq \frac{1}{2}$ + an integer. If $u \in Z^{s'}(I^n)$ has the property that for each positive integer p there exists $\varphi_p \in P_p(I^n)$ satisfying

$$\|u-\varphi_p\|_{Z^{s'}(I^n)} \leq C^* p^{-s+s'} \tag{2.7}$$

with C^* independent of p, then $u \in Z^{s-\varepsilon}(I^n)$ for arbitrarily small $\varepsilon > 0$.

Pf: The main part of the proof consists of showing that if u satisfies the hypothesis of the theorem, then u belongs to

the real interpolation space $(Z^{s'}(I^n), Z^k(I^n))_{\frac{s-s'}{k-s'}-\varepsilon,2}$ for all $k > s$ and all sufficiently small $\varepsilon > 0$. The result then follows from part (iii) of Theorem 2.1.

Let $u = \sum_{|\underline{m}|=0}^{\infty} a_{\underline{m}}\Phi_{\underline{m}} \in L_2(I^n)$ satisfy (2.7). It follows that for each $p > 0$,

$$\|u - \sum_{0\le|\underline{m}|\le p} a_{\underline{m}}\Phi_{\underline{m}}\|_{Z^{s'}(I^n)} \le C^* p^{-s+s'} .$$

Let $k > s$ and for each $j = 1,2,\ldots$ set $u_j = \sum_{0\le|\underline{m}|\le 2^j} a_{\underline{m}}\Phi_{\underline{m}}$. Since each u_i with $i \ge 2$ may be written as $u_i = u_1 + \sum_{j=2}^{i} (u_j - u_{j-1})$, one obtains that

$$\|u_i\|_{Z^k(I^n)} \le \|u_1\|_{Z^k(I^n)} + \sum_{j=2}^{i} \|u_j - u_{j-1}\|_{Z^k(I^n)} .$$

Now,

$$\begin{aligned}
\|u_1\|_{Z^k(I^n)} &\le C \left(\sum_{0\le|\underline{m}|\le 2} a_{\underline{m}}^2 [1 + \sum_{i=1}^{n} m_i^{2k}] \right)^{1/2} \\
&\le C \left(\sum_{0\le|\underline{m}|\le 2} a_{\underline{m}}^2 [1 + \sum_{i=1}^{n} m_i^{2s'}] \right)^{1/2} \\
&\le C\|u\|_{Z^{s'}(I^n)}
\end{aligned}$$

and

$$\|u_j - u_{j-1}\|_{Z^k(I^n)} \le C \left(\sum_{2^{j-1}<|\underline{m}|\le 2^j} a_{\underline{m}}^2 [1 + \sum_{i=1}^{n} m_i^{2k}] \right)^{1/2}$$

$$\leq C\,2^{(k-s')j}\left(\sum_{2^{j-1}<|\underline{m}|\leq 2^j} |a_{\underline{m}}|^2 [1 + \sum_{i=1}^{n} m_i^{2s'}]\right)^{1/2}$$

$$\leq C\,2^{(k-s')j}\|u_j-u_{j-1}\|_{Z^{s'}(I^n)}.$$

Therefore,

$$\|u_i\|_{Z^k(I^n)} \leq C(\|u\|_{Z^{s'}(I^n)} + \sum_{j=2}^{i} 2^{(k-s')j}\|u_j-u_{j-1}\|_{Z^{s'}(I^n)}).$$

Since

$$\|u_j-u_{j-1}\|_{Z^{s'}(I^n)} \leq \|u_j-u\|_{Z^{s'}(I^n)} + \|u-u_{j-1}\|_{Z^{s'}(I^n)} \leq C^* 2^{(-s+s')j},$$

it follows that

$$\|u_i\|_{Z^k(I^n)} \leq C(\|u\|_{Z^{s'}(I^n)} + \sum_{j=2}^{i} 2^{(k-s)j}$$

$$\leq C(\|u\|_{Z^{s'}(I^n)} + 2^{(k-s)i}).$$

For $t > 0$, consider

$$K(u,t) = \inf_{u=v+w} (\|v\|_{Z^{s'}(I^n)} + t\|w\|_{Z^k(I^n)}).$$

Taking $v = u - u_i$ and $w = u_i$, one obtains that

$$K(u,t) \leq C(C^* 2^{(-s+s')i} + t\|u\|_{Z^{s'}(I^n)} + t2^{(k-s)i})$$

$$\leq C(\|u\|_{Z^{s'}(I^n)} + C^*)(t2^{(k-s)i} + 2^{(-s+s')i}).$$

For $0 < t < 1$, choose i so that $2^{i} \le t^{\frac{-1}{k-s'}} < 2^{i+1}$. Then $2^{(-s+s')i} < 2^{(s-s')} t^{\frac{s-s'}{k-s'}}$ and $2^{(k-s)i} = 2^{(k-s')i(1-\frac{s-s'}{k-s'})}$ $t^{\frac{s-s'}{k-s'}-1}$. Therefore, for $0 < t < 1$,

$$K(u,t) \le C(\|u\|_{Z^{s'}(I^n)} + C^*)t^{\frac{s-s'}{k-s'}} .$$

For $t \ge 1$, take $v = u$ and $w = 0$ to obtain

$$K(u,t) \le C\|u\|_{Z^{s'}(I^n)} .$$

Suppose that $0 < \varepsilon < \frac{s-s'}{k-s'}$. Then

$$\int_0^\infty (t^{-\frac{s-s'}{k-s'}+\varepsilon} K(u,t))^2 \frac{dt}{t} \le C(\|u\|_{Z^{s'}(I^n)} + C^*)^2 \int_0^1 t^{2\varepsilon-1}\, dt$$

$$+ C\|u\|^2_{Z^{s'}(I^n)} \int_1^\infty t^{-2(\frac{s-s'}{k-s'})+2\varepsilon-1}\, dt$$

$$\le C(\varepsilon)(\|u\|_{Z^{s'}(I^n)} + C^*)^2$$

and thus $u \in (Z^{s'}(I^n), Z^k(I^n))_{\frac{s-s'}{k-s'}-\varepsilon,2}$, which completes the proof.

Let Ω be a domain in $\mathbb{R}^n$ such that there exists a triangulation Δ of Ω into open n-simplices Ω_i, $1 \le i \le M$. Let σ^ν, $\nu = 1,\ldots,N$, denote the vertices of Δ. Since it will be convenient to be able to refer to the vertices of a

particular simplex Ω_i, for $1 \le i \le M$ let $\sigma_{i,j}$, $1 \le j \le n+1$, denote the vertices of Ω_i. Hence, if a vertex σ^ν of Δ is also a vertex of Ω_i, then $\sigma^\nu = \sigma_{i,j}$, for some j, $1 \le j \le n+1$. It will also be useful to define the index set $S_\Delta = \{(i,j): 1 \le i \le M,\ 1 \le j \le n+1\}$.

Consider a simplex $\Omega_i \in \Delta$ and one of its vertices $\sigma_{i,j}$, $1 \le j \le n+1$. Letting $e_1,\ldots,e_n$ be vectors based at $\sigma_{i,j}$ and terminating at the other n vertices of Ω_i, define in $\mathbb{R}^n$ the parallelepiped

$$\Omega_{i,j} = \{x \in \mathbb{R}^n: x = \sigma_{i,j} + \sum_{m=1}^{n} t_m e_m, \quad 0 < t_m < 1\}.$$

Clearly, $\Omega_i \subset \Omega_{i,j}$ for all $j = 1,\ldots,n+1$. For each $(i,j) \in S_\Delta$, choose an affine mapping $T_{i,j}$ in $\mathbb{R}^n$ which maps $\Omega_{i,j}$ onto I^n and $\sigma_{i,j}$ onto the point $(1,\ldots,1)$.

Let $\{\eta_\nu\}_{\nu=1}^{N}$ be a smooth partition of unity on $\bar{\Omega}$ such that for each $\nu = 1,\ldots,N$ supp η_ν contains the vertex σ^ν and supp η_ν intersects only those closed simplices $\bar{\Omega}_i$ which have σ_ν as a vertex. For any $u \in L_2(\Omega)$ and $(i,j) \in S_\Delta$, define on $\Omega_{i,j}$

$$\text{(2.8)} \quad u_{i,j} = \begin{cases} u\eta_\nu & \text{in } \bar{\Omega}_i \text{ where } \nu \text{ is such that } \sigma_{i,j} = \sigma^\nu \\ 0 & \text{in } \overline{\Omega_{i,j} \setminus \Omega_i}\,. \end{cases}$$

By the assumptions on η_ν, one observes that $u_{i,j} = u$ in a neighborhood of $\sigma_{i,j}$ and $u_{i,j} \equiv 0$ outside of a neighborhood U of $\sigma_{i,j}$ such that $U \cap \overline{\Omega_{i,j} \setminus \Omega_i} = \emptyset$.

For each real [illegible] ≥ 0 such that [illegible] $\neq \frac{1}{2}$ + an integer, let

$$Z^s(\Omega;\Delta) = \{u: \|u\|_{Z^s(\Omega;\Delta)} < \infty\}$$

where

$$\|u\|_{Z^s(\Omega;\Delta)} = \left(\sum_{(i,j)\in S_\Delta} \|u_{i,j}\circ T_{i,j}^{-1}\|^2_{Z^s(I^n)}\right)^{1/2}.$$

For each real $s \geq 0$, let $H^s(\Omega_i)$, $1 \leq i \leq M$, denote the usual Sobolev space of order s on Ω_i and set

$$H^s(\Omega;\Delta) = \{u: \|u\|_{H^s(\Omega;\Delta)} < \infty\}$$

where

$$\|u\|_{H^s(\Omega;\Delta)} = \left(\sum_{i=1}^{M} \|u\|^2_{H^s(\Omega_i)}\right)^{1/2}.$$

For each non-negative integer p, let $P_p(\Omega_i)$, $1 \leq i \leq M$, denote the space of all polynomials on Ω_i of degree at most p. Finally, let

$$P_p(\Omega;\Delta) = \{u: u|_{\Omega_i} \in P_p(\Omega_i), \quad 1 \leq i \leq M\}$$

<u>Theorem 2.4</u>. Let s and s' be such that $s > 2s' \geq 0$ and $s, 2s' \neq \frac{1}{2}$ + an integer. If $u \in Z^s(\Omega;\Delta)$ then for each non-negative integer p there exists $\varphi_p \in P_p(\Omega;\Delta)$ such that

$$\|u-\varphi_p\|_{H^s(\Omega;\Delta)} \leq Cp^{-s+2s'}\|u\|_{Z^s(\Omega;\Delta)} \tag{2.9}$$

where $C = C(s,s')$ is independent of u and p.

<u>Pf</u>: For each $(i,j) \in S_\Delta$, Theorem 2.[illegible] yields $\varphi_p^{i,j} \in P_p(I^n)$ satisfying

$$(2.10) \quad \|u_{i,j} \circ T_{i,j}^{-1} - \varphi_p^{i,j}\|_{Z^{2s'}(I^n)} \leq Cp^{-s+2s'}\|u_{i,j} \circ T_{i,j}^{-1}\|_{Z^s(I^n)}.$$

Setting*

$$\varphi_p(x) = \sum_{j=1}^{n+1} \varphi_p^{i,j} \circ T_{i,j}(x) \quad \text{for } x \in \Omega_i, \quad 1 \leq i \leq M,$$

(2.9) follows from Lemma 2.6, (2.10), and the triangle inequality.

For the special case $s' = 0$, the following result is the inverse of Theorem 2.4, up to an arbitrarily small $\varepsilon > 0$.

<u>Theorem 2.5</u>. Let s be any non-negative real number such that $s \neq \frac{1}{2}$ + an integer. If $u \in L_2(\Omega)$ has the property that, for each non-negative integer p, there exists $\varphi_p \in P_p(\Omega;\Delta)$ satisfying

$$\|u-\varphi_p\|_{L_2(\Omega)} \leq Cp^{-s}$$

with C independent of p, then $u \in Z^{s-\varepsilon}(\Omega;\Delta)$ for arbitrarily small $\varepsilon > 0$.

---

*Here and frequently in the remainder of the paper, implicit use is made of the fact that the composition of a polynomial of degree $p \ge 1$ with an affine mapping is again a polynomial of degree p.

If: Let Q_k, $k = 1,\dots,K$ be a collection of open cubes in $\mathbb{R}^n$ such that $\sum_{k=1}^{K} Q_k = I^n$ and such that the faces of each of the cubes Q_k are parallel to the faces of I^n, i.e., each is of the form

$$Q_k = \{x \in I^n : -1 \le a_m^k < x_m < b_m^k \le 1, \ 1 \le m \le n\}.$$

For each $k = 1,\dots,K$, let R_k denote an affine mapping in $\mathbb{R}^n$ such that $R_k(Q_k) = I^n$.

Fix $(i,j) \in S_\Delta$ and let ν be such that $\sigma^\nu = \sigma_{i,j}$. One may assume that the cubes Q_k have been chosen sufficiently small so that if $T_{i,j}^{-1}(Q_k) \cap \operatorname{supp} \eta_\nu \neq \emptyset$ for some k, then $T_{i,j}^{-1}(Q_k) \subset \&_i$. Given any such k, it follows from the hypothesis that for each non-negative integer p there exists $\psi_p \in P_p(I^n)$ satisfying

$$\|u \circ T_{i,j}^{-1} \circ R_k^{-1} - \psi_p\|_{L_2(I^n)} \le Cp^{-s}.$$

By Theorem 2.3, this implies that $u \circ T_{i,j}^{-1} \circ R_k^{-1} \in Z^{s-\varepsilon}(I^n)$ for arbitrarily small $\varepsilon > 0$. Hence, since η_ν is smooth, one obtains that if k is such that $T_{i,j}^{-1}(Q_k) \cap \operatorname{supp} \eta_\nu \neq \emptyset$, then $(u\eta_\nu) \circ T_{i,j}^{-1} \circ R_k^{-1} \in Z^{s-\varepsilon}(I^n)$. Furthermore, if k is such that $T_{i,j}^{-1}(Q_k) \cap \operatorname{supp} \eta_\nu = \emptyset$, then $(u\eta_\nu) \circ T_{i,j}^{-1} \circ R_k^{-1} = 0$ on I^n. Letting $\{\chi_k\}_{k=1}^{K}$ denote a smooth partition of unity subordinate to the open cover $\{Q_k\}$, it is not difficult to show that

$$\|(u\eta_\nu)\circ T_{i,j}^{-1}\|_{H^{s-\varepsilon}(\mathbb{R}^n)} \le \sum_{k=1}^{r} \|\lambda_k (u\eta_\nu)\circ T_{i,j}^{-1}\|_{H^{s-\varepsilon}(\mathbb{R}^n)}$$

$$\le C \sum_{k=1}^{r} \|(u\eta_\nu)\circ T_{i,j}^{-1}\circ R_k^{-1}\|_{H^{s-\varepsilon}(\mathbb{R}^n)} .$$

Hence, $(u\eta_\nu)\circ T_{i,j}^{-1} \in H^{s-\varepsilon}(\mathbb{R}^n)$, which completes the proof.

3. Approximation by conforming piecewise polynomials on triangulated domains

In the present and following sections, Theorems 2.4 and 2.5 are extended to obtain similar results for approximation by conforming piecewise polynomials on triangulated domains in $\mathbb{R}^n$. The essence of these results is that, up to an arbitrarily small $\varepsilon > 0$, one can obtain conforming piecewise polynomials yielding the same degree of approximation as the non-conforming piecewise polynomials of Theorem 2.4 provided that the function being approximated satisfies the same compatibility conditions across the common boundaries of adjacent simplices.

Let Ω denote a domain of $\mathbb{R}^n$ such that there exists a triangulation Δ of Ω into simplices Ω_i, $i = 1,\ldots,M$. Recalling the spaces $Z^s(\Omega;\Delta)$ and $P_p(\Omega;\Delta)$ defined in section 2, for each non-negative integer ℓ and p set

$$
(3.1)\qquad
\begin{aligned}
Z^s_\ell(\Omega;\Delta) &= Z^s(\Omega;\Delta) \cap C^\ell(\bar{\Omega}) \\
P^\ell_p(\Omega;\Delta) &= P_p(\Omega;\Delta) \cap C^\ell(\bar{\Omega})
\end{aligned}
$$

where $C^\ell(\bar{\Omega})$ denotes the set of all functions which along with their first ℓ derivatives are continuous on $\bar{\Omega}$. It is clear that $P^\ell_p(\Omega;\Delta)$ is precisely the set of all functions in $P_p(\Omega;\Delta)$ which along with their first ℓ derivatives are continuous across the common boundaries of adjacent simplices of Δ. As a consequence of the following lemma, if ℓ is any integer satisfying $0 \le \ell < \frac{s-n}{2}$, then it similarly holds that $Z^s_\ell(\Omega;\Delta)$ is the subspace of all functions in $Z^s(\Omega;\Delta)$ which along with their first ℓ derivatives are continuous across the common

boundaries of adjacent simplices of Δ.

<u>Lemma 3.1</u>. Let s be a positive real number such that $s \neq \frac{1}{2}$ + an integer, and let ℓ be any integer satisfying $0 \le \ell < \frac{s-n}{2}$. Then $Z^s(I^n)$ is continuously embedded in $C^\ell(\bar{I}^n)$.

<u>Pf</u>: By the Sobolev Lemma [10], it holds that $H^{s/2}(I^n)$ continuously imbeds in $C^\ell(\bar{I}^n)$ provided that $s - 2\ell > n$. The result then follows from Lemma 2.6.

In addition to the definitions (3.1), it is convenient to set $Z^s_{-1}(\Omega;\Delta) = Z^s(\Omega;\Delta)$.

<u>Theorem 3.1</u>. Let s and s' be such that $s > 2s' \ge 0$ and $s, 2s' \neq \frac{1}{2}$ + an integer. Let ℓ^* be the largest integer strictly less than $\frac{s-n}{2}$. If $u \in Z^s_\ell(\Omega;\Delta)$ for some integer ℓ, $0 \le \ell < \ell^*$, then for each non-negative integer p there exists $\varphi_p \in P^\ell_p(\Omega;\Delta)$ such that for arbitrarily small $\varepsilon > 0$,

$$(3.2) \qquad \|u-\varphi_p\|_{H^{s'}(\Omega;\Delta)} \le Cp^{-s+2s'+\varepsilon} \|u\|_{Z^s(\Omega;\Delta)}$$

where $C = C(s,s',\varepsilon)$ is independent of u and p. Moreover, if $u \in Z^s_{\ell^*}(\Omega;\Delta)$ then for any non-negative integers ℓ and p there exists $\varphi_p \in P^\ell_p(\Omega;\Delta)$ such that (3.2) holds.

The following inverse result says that in the case $s' = 0$, Theorem 3.1 is the best result possible, up to an arbitrarily small $\varepsilon > 0$.

<u>Theorem 3.2</u>. Let s be any positive real number such that $s \neq \frac{1}{2}$ + an integer and let ℓ^* be the largest integer strictly

less than $\frac{s-n}{2}$. Suppose that u and ℓ are such that for each non-negative integer p there exists $\varphi_p \in P_p^{\ell}(\Omega;\Delta)$ satisfying

$$\|u-\varphi_p\|_{L_2(\Omega)} \le Cp^{-s} \tag{3.3}$$

for some constant C independent of p. Then $u \in Z^{s-\varepsilon}_{\min(\ell,\ell^*)}(\Omega;\Delta)$ for arbitrarily small $\varepsilon > 0$.

The basic idea behind the proof of Theorem 3.1 is to carefully modify the piecewise polynomials produced by Theorem 3.4 in such a way as to achieve the required amount of regularity across the common boundaries of adjacent simplices of Δ without degrading the degree of approximation by more than an arbitrarily small amount ε. The technique is most clearly observed in the case $n = 1$ which is treated separately in section 4. The proofs for the cases $n > 1$ involve some additional technicalities, the nature of which is exemplified by the proof for $n = 2$ in section 5. Although the proofs for $n > 2$ are not given here, it will be clear from the cases considered that these may be obtained by similar arguments.

In order to help simplify the exposition, the important issue of boundary conditions has been neglected. However, it will be easily seen that the same techniques which allow one to construct piecewise polynomials with ℓ continuous derivatives across the common boundaries of adjacent simplices may also be used to enforce any homogeneous boundary conditions satisfied by the approximated function and its first ℓ derivatives on the

In the error analysis of the p-version of the finite element method one needs approximation results in the Sobolev spaces $H^{s'}(\Omega)$ rather than the "piecewise" Sobolev spaces $H^{s'}(\Omega;\Lambda)$. It still remains to show how such results are obtained. The following Lemma is the key.

<u>Lemma</u> 3.2. Let s be any positive real number such that $s \neq \frac{1}{2}$ + an integer. If s' is such that $s > 2s' \geq 0$ and if ℓ is an integer such that $s' - \frac{3}{2} < \ell < \frac{s-n}{2}$, then $Z^s_\ell(\Omega;\Lambda)$ continuously imbeds in $H^{s'}(\Omega)$.

<u>Pf</u>: It is a simple consequence of the Sobolev trace theory [10] that if $u \in H^{s'}(\Omega;\Lambda)$ and if $u \in C^\ell(\bar{\Omega})$, then $u \in H^{s'}(\Omega)$ provided that $\ell > s' - \frac{3}{2}$. The result then follows from Lemma 3.1.

Theorem 3.1 together with Lemma 3.2 yields the following:

<u>Theorem 3.3</u>. Let s and s' be such that $s > 2s' \geq 0$ and $s, 2s' \neq \frac{1}{2}$ + an integer. Let ℓ^* be the largest integer strictly less than $\frac{s-n}{2}$. If $u \in Z^s_\ell(\Omega;\Lambda)$ for some integer ℓ satisfying $s' - \frac{3}{2} < \ell < \ell^*$, then for each non-negative integer p there exists $\varphi_p \in P^\ell_p(\Omega;\Lambda)$ such that for arbitrarily small $\varepsilon > 0$,

$$\|u-\varphi_p\|_{H^{s'}(\Omega)} \leq Cp^{-s+2s'+\varepsilon}\|u\|_{Z^s(\Omega;\Lambda)} \tag{3.4}$$

where $C = C(s,s',\varepsilon)$ is independent of u and p. Moreover, if $u \in Z^s_{\ell^*}(\Omega;\Lambda)$ then for any integer $\ell > s' - \frac{3}{2}$ and for

any non-negative integer ; there exists [illegible] $P_j^i(\Omega;A)$ [illegible]
that (3.4) holds.

4. Approximation by conforming piecewise polynomials continued: the case $n = 1$

The proofs of Theorems 3.1 and 3.2 in the case $n = 1$ will follow a number of technical lemmas.

Lemma 4.1. Let s be a non-negative real number. Then for any $\varphi_p \in P_p(I)$,

$$(4.1) \qquad \|\varphi_p\|_{H^s(I)} \le C(s)p^{2s}\|\varphi_p\|_{L_2(I)}$$

where C is independent of p and φ_p.

Pf: By Schmidt's inequality (see e.g. [5]), it holds that for any $\varphi_p \in P_p(I)$,

$$\int_I [\frac{d\varphi_p}{dx}(x)]^2\,dx \le \frac{(p+1)^4}{2}\int_I \varphi_p^2(x)dx$$

and (4.1) follows by induction for s an integer. A standard interpolation argument yields (4.1) for non-integer s.

As in section 2, let $\{P_n\}$ denote the system of Legendre polynomials on I normalized so that $\|P_n\|_{L_2(I)} = 1$ for all n.

Lemma 4.2. For each non-negative integer n,

$$\frac{d^k}{dx^k}P_n(\pm 1) = (\pm 1)^n \frac{(2n+1)^{1/2}}{2^{k+\frac{1}{2}}k!}\prod_{\mu=-k+1}^{k}(n+\mu), \qquad k = 0,\ldots,n.$$

Pf: This follows immediately from equations 22.2.1, 22.4.1, 22.5.27, and 22.5.37 of [1], taking into account the normalization $\|P_n\|_{L_2(I)} = 1$.

Lemma 4.3. Let k be a non-negative integer and let s be any non-negative real number such that $s \neq \frac{1}{2} +$ an integer. For each integer $p > k + 1$ there exists $\varphi_p \in P_p(I)$ such that

$$\varphi_p^{(m)}(1) = \begin{cases} 1 & \text{if} \quad m = k \\ 0 & \text{if} \quad 0 \leq m < k \end{cases} \tag{4.2}$$

and

$$\|\varphi_p\|_{Z^s(I)} \leq Cp^{-(2k+1)+s} \tag{4.3}$$

where C is independent of p.

Pf: Fix $p \geq k + 1$ and consider the following optimization problem: Minimize the quadratic objective function $\sum_{n=k}^{p} a_n^2$ over all $(p-k+1)$-tuples $(a_k,\ldots,a_p)$ satisfying the linear constraints

$$\sum_{n=k}^{p} a_n P_n^{(m)}(1) = \begin{cases} 1, & m = k \\ 0, & 0 \leq m < k. \end{cases}$$

If it can be shown that there exists a solution $(a_k,\ldots,a_p)$ such that

$$\sum_{n=k}^{p} a_n^2 \leq Cp^{-2(2k+1)} \tag{4.4}$$

where C is independent of p, then by setting $\varphi_p = \sum_{n=k}^{p} a_n P_n$ it follows that

$$\|\varphi_p\|^2_{H^s(I)} \le C \sum_{n=k}^{p} a_n^2 n^{2s} \le Cp^{2s} \sum_{n=k}^{p} a_n^2$$

$$\le Cp^{-2(2k+1)+2s},$$

and φ_p satisfies (4.2) and (4.3).

Applying the method of Lagrange multipliers, one seeks a stationary point of the function

$$\Phi(a_k,\ldots,a_p,\lambda_0,\ldots,\lambda_k)$$

$$= \sum_{n=k}^{p} a_n^2 - \sum_{m=0}^{k-1} \lambda_m \Big[\sum_{n=k}^{p} a_n P_n^{(m)}(1) \Big] - \lambda_k \Big[\sum_{n=k}^{p} a_n P_n^{(k)}(1) - 1 \Big].$$

Setting $\dfrac{\partial \Phi}{\partial a_n} = 0$ for $n = k,\ldots,p$, applying Lemma 4.2, and solving for a_n, it follows that

$$(4.5)\qquad a_n = \frac{1}{2}\Big(\frac{2n+1}{2}\Big)^{1/2} \sum_{m=0}^{k} \frac{\lambda_m}{2^m m!} \prod_{\mu=-m+1}^{m} (n+\mu), \qquad k \le n \le p.$$

Furthermore, setting $\dfrac{\partial \Phi}{\partial \lambda_\ell} = 0$ for $\ell = 0,\ldots,k$ and again applying Lemma 4.2, one obtains

$$(4.6)\qquad \begin{aligned} \sum_{n=k}^{p} a_n (2n+1)^{1/2} \prod_{\mu=-\ell+1}^{\ell} (n+\mu) &= 0, \qquad 0 \le \ell < k, \\ \sum_{n=k}^{p} a_n (2n+1)^{1/2} \prod_{\mu=-k+1}^{k} (n+\mu) &= 2^{k+\frac{1}{2}} k!\,. \end{aligned}$$

The substitution of (4.5) into (4.6) then yields

$$(4.7)\qquad \begin{aligned} &\sum_{m=0}^{k} \psi_{\ell,m}(p)\,\frac{\lambda_m}{2^m m!} = 0, \qquad 0 \le \ell < k, \\ &\sum_{m=0}^{k} \psi_{\ell,k}(p)\,\frac{\lambda_m}{2^m m!} = -p^{k+2}\,k!, \end{aligned}$$

where

$$\begin{aligned} \psi_{\ell,m}(p) &= \sum_{n=k}^{p} \Big[\prod_{\mu=-m+1}^{m}(n+\mu)\Big](2n+1)\prod_{\mu=-\ell+1}^{\ell}(n+\mu) \\ &= \frac{1}{\ell+m+1}\,p^{2(\ell+m+1)} + \text{a polynomial in } p \\ &\qquad \text{of degree less than } 2(\ell+m+1). \end{aligned}$$

Regarding (4.7) as a linear system to be solved for the quantities $\lambda_m/2^m m!$, $0 \le m \le k$, one observes that if $D(p)$ is the determinant of the coefficient matrix $(\psi_{\ell,m}(p))$, then

$$\begin{aligned} D(p) &= \beta_{k+1}\,p^{2(k+1)^2} + \text{a polynomial in } p \\ &\qquad \text{of degree less than } 2(k+1)^2 \end{aligned}$$

where β_{k+1} is the determinant of the $(k+1)^{st}$ principal minor of the Hilbert matrix. Since $\beta_{k+1} \neq 0$, (4.7) may be uniquely solved for the $\lambda_m/2^m m!$ provided that p is greater than the largest root of $D(p)$ (which depends only on k). Applying Cramer's rule, it follows that for $m = 0,\ldots,k$,

$$(4.8)\qquad \begin{aligned} \frac{\lambda_m}{2^m m!} &= D(p)^{-1}\cdot(\text{a polynomial in } p \text{ of degree } 2(k+1)^2-2(m+k+1)) \\ &= O(p^{-2(m+k+1)}) \quad \text{as } p \to \infty. \end{aligned}$$

Hence, for $m = 0,\ldots,k$ and $k \le n \le p$, one obtains from (4.5) and (4.8) that

$$|a_n| \le C(2n+1)^{1/2}\, p^{-2(k+1)} \sum_{m=0}^{k} p^{-2m} \prod_{\mu=-m+1}^{m} (n+\mu)$$

$$\le Cp^{-2k-\frac{3}{2}}.$$

Since this implies (4.4), the proof is complete.

<u>Lemma 4.4</u>. Let ℓ be a non-negative integer and let α_k, β_k, $k = 0,\ldots,\ell$, be real numbers. Let s be any non-negative real number such that $s \neq \frac{1}{2}$ + an integer. Then, for any integer $p \ge 2\ell + 2$ there exists $\psi_p \in P_p(I)$ such that

$$\text{(4.9)}\qquad \begin{aligned} \psi_p^{(k)}(1) &= \alpha_k, \quad 0 \le k \le \ell \\ \psi_p^{(k)}(-1) &= \beta_k, \quad 0 \le k \le \ell \end{aligned}$$

and for arbitrarily small $\varepsilon > 0$,

$$\text{(4.10)}\qquad \|\psi_p\|_{Z^s(I)} \le C_1(|\alpha_\ell|+|\beta_\ell|)p^{-(2\ell+1)+s} + C_2(\varepsilon) \sum_{k=0}^{\ell-1} (|\alpha_k|+|\beta_k|)p^{-(2k+1)+s+\varepsilon}$$

where C_1 and C_2 are independent of p. If $\ell = 0$ the second term in the right-hand side of (4.10) is omitted.

Proof: It suffices to prove the result for $\beta_k = 0$, $0 \le k \le \ell$, since the general result may then be obtained via superposition.

By Lemma 4.3, for each k, $0 \le k \le \ell$, and each integer $\bar{p} \ge \ell + 1$, there exists $\varphi_{\bar{p},k} \in P_{\bar{p}}(I)$ such that

$$\varphi^{(m)}_{\bar{p},k}(1) = \begin{cases} 1 & \text{if} \quad m = k \\ 0 & \text{if} \quad 0 \le m < k, \end{cases}$$

(4.11)

$$\|\varphi_{\bar{p},k}\|_{Z^{s'}(I)} \le C\bar{p}^{-(2k+1)+s'}$$

for any real $s' \ge 0$ such that $s' \ne \frac{1}{2}$ + an integer. Set $p = \bar{p} + \ell + 1$ and define

$$\psi_{p,0}(x) = \alpha_0\varphi_{\bar{p},0}(x)\left(\frac{x+1}{2}\right)^{\ell+1}, \qquad x \in I.$$

If $\ell > 0$ then for $k = 1,\ldots,\ell$ recursively define

$$\psi_{p,k}(x) = (\alpha_k - \psi^{(k)}_{p,k-1}(1))\varphi_{\bar{p},k}(x)\left(\frac{x+1}{2}\right)^{\ell+1} + \psi_{p,k-1}(x), \qquad x \in I.$$

Setting $\psi_p = \psi_{p,\ell}$ one checks that ψ_p satisfies (4.9), and if $\ell = 0$, (4.11) implies that

$$\|\psi_p\|_{Z^s(I)} \le C|\alpha_0|\bar{p}^{-1+s} \le C|\alpha_0|p^{-1+s}$$

which establishes (4.10) for $\ell = 0$. If $\ell > 0$, then the Sobolev Lemma [10] together with Lemma 4.1 yields that for arbitrarily small $\varepsilon > 0$,

$$|\psi^{(k)}_{p,k-1}(1)| \le C(\varepsilon)\|\psi_{p,k-1}\|_{H^{k+\frac{1}{2}+\varepsilon}(I)}$$

$$\le C(\varepsilon)p^{2k+1+2\varepsilon}\|\psi_{p,k-1}\|_{L_2(I)}.$$

Hence, for $k = 1, \ldots, \ell$ and any $s' \geq 0$ such that $s' \neq \frac{1}{2}$ + an integer,

$$\|\psi_{p,k}\|_{Z^{s'}(I)} \leq C|a_k| p^{-(2k+1)+s'} + C(\varepsilon) p^{2k+1+2\varepsilon} p^{-(2k+1)+s'} \tag{4.12}$$
$$\cdot \|\psi_{p,k-1}\|_{L_2(I)} + \|\psi_{p,k-1}\|_{L_2(I)}$$
$$\leq C|a_k| p^{-(2k+1)+s'} + C(\varepsilon) p^{s'+2\varepsilon} \|\psi_{p,k-1}\|_{L_2(I)}.$$

By applying (4.12) with $s' = s$ for $k = \ell$ and then successively with $s' = 0$ for $k = \ell - 1, \ldots, 0$, (4.10) follows.

<u>Pf. of Theorem 3.1</u>. Recalling the notation of section 2, fix $(i,j) \in S_\Delta$ and consider $u_{i,j} \in Z^s(I)$. Expand $u_{i,j}$ in its Legendre series $\sum_{n=0}^{\infty} a_n P_n$ and set $\xi_p = \sum_{n=0}^{p} a_n P_n$ for each non-negative integer p. By Theorem 2.2,

$$\|u_{i,j} - \xi_p\|_{Z^{2s'}(I)} \leq C p^{-(s-2s')} \|u_{i,j}\|_{Z^s(I)}. \tag{4.13}$$

By Lemma 3.1 together with (4.13), it follows that, for $0 \leq k < \frac{s-1}{2}$ and arbitrarily small $\varepsilon > 0$,

$$|u_{i,j}^{(k)}(\pm 1) - \xi_p^{(k)}(\pm 1)| \leq C(\varepsilon) \|u_{i,j} - \xi_p\|_{Z^{2k+1+\varepsilon}(I)} \tag{4.14}$$
$$\leq C(\varepsilon) p^{-s+2k+1+\varepsilon} \|u_{i,j}\|_{Z^s(I)}.$$

If $\frac{s-1}{2} \leq k \leq \ell$, then Lemma 3.1 implies that again for arbitrarily small $\varepsilon > 0$,

$$
\begin{aligned}
|\xi_p^{(k)}(\pm 1)| &\le C(\varepsilon)\, \|\xi_p\|_{Z^{2k+1+\varepsilon}(I)} \\
&\le C(\varepsilon)\Big(\sum_{n=0}^{\infty} a_n^2\, n^{2s}\, n^{-2s+4k+2+2\varepsilon}\Big)^{1/2} \\
&\le C(\varepsilon) p^{-s+2k+1+\varepsilon} \|u_{i,j}\|_{Z^s(I)} .
\end{aligned}
$$

For each $p \ge 2\ell + 2$, one obtains from Lemma 4.4 that there exists $\psi_p \in P_p(I)$ such that

$$
\psi_p^{(k)}(\pm 1) = \begin{cases} u_{i,j}^{(k)}(\pm 1) - \xi_p^{(k)}(\pm 1) & \text{if } 0 \le k < \frac{s-1}{2} \\ -\xi_p^{(k)}(\pm 1) & \text{if } \frac{s-1}{2} \le k \le \ell, \end{cases}
$$

and such that for arbitrarily small $\varepsilon > 0$,

(4.15)
$$
\begin{aligned}
\|\psi_p\|_{Z^{2s'}(I)} &\le C(\varepsilon) \sum_{k=0}^{\ell} (|\psi_p^{(k)}(1)| + |\psi_p^{(k)}(-1)|) p^{-(2k+1)+2s'+\varepsilon} \\
&\le C(\varepsilon) p^{-s+2s'+\varepsilon} \|u_{i,j}\|_{Z^s(I)} .
\end{aligned}
$$

Set $\varphi_{p,i,j} = \xi_p + \psi_p$. Then by (4.13), (4.15), and Lemma 2.3,

$$
\begin{aligned}
\|u_{i,j} - \varphi_{p,i,j}\|_{H^{s'}(I)} &\le \|u_{i,j} - \xi_p\|_{H^{s'}(I)} + \|\psi_p\|_{H^{s'}(I)} \\
&\le C(\|u_{i,j} - \xi_p\|_{Z^{2s'}(I)} + \|\psi_p\|_{Z^{2s'}(I)}) \\
&\le C(\varepsilon) p^{-s+2s'+\varepsilon} \|u_{i,j}\|_{Z^s(I)} ,
\end{aligned}
$$

and

$$\rho^{(k)}_{p,i,j}(\pm 1) = \begin{cases} u^{(k)}_{i,j}(\pm 1) & \text{if } 0 \le k < \frac{s-1}{2}, \\ 0 & \text{if } \frac{s-1}{2} \le k \le \ell. \end{cases}$$

Assuming that the above has been carried out for each $(i,j) \in S_\Delta$, the desired $\varphi_p \in P^\ell_p(\Omega;\Delta)$ is given by

$$\varphi_p|_{\Omega_i} = \sum_{j=1}^{2} \rho_{p,i,j} \circ T_{i,j}, \qquad 1 \le i \le M.$$

Pf. of Theorem 3.2. It follows immediately from Theorem 2.5 that $u \in Z^{s-\varepsilon}(\Omega;\Delta) = Z^{s-\varepsilon}_{-1}(\Omega;\Delta)$ for arbitrarily small $\varepsilon > 0$, so it only remains to check the regularity of u at the nodal points of the subdivision Δ.

To this end, let Ω_1 and Ω_2 be adjacent intervals of Δ with common endpoint σ, and let $u_i = u|_{\Omega_i}$, $i = 1,2$. Since $u \in Z^{s-\varepsilon}(\Omega;\Delta)$, it follows as in (4.13),(4.14) that for each non-negative integer p there exists $\psi_{p,i} \in P_p(\Omega_i)$ such that

$$\|u_i - \psi_{p,i}\|_{L_2(\Omega_i)} \le C(\varepsilon) p^{-s+\varepsilon} \|u\|_{Z^s(\Omega;\Delta)}, \tag{4.16}$$

and for $0 \le k \le \ell^*$,

$$|u_i^{(k)}(\sigma) - \psi_{p,i}^{(k)}(\sigma)| \le C(\varepsilon) p^{-s+2k+1+\varepsilon} \|u\|_{Z^s(\Omega;\Delta)}. \tag{4.17}$$

Letting $\varphi_{p,i} = \varphi_p|_{\Omega_i}$, $i = 1,2$, it follows from the hypothesis that $\varphi^{(k)}_{p,1}(\sigma) = \varphi^{(k)}_{p,2}(\sigma)$, $0 \le k \le \ell$, and

$$\|u_i-\varphi_{p,i}\|_{L_2(\Omega_i)} \leq Cp^{-s}, \qquad i = 1,2. \tag{4.18}$$

By the Sobolev Lemma together with Lemma 4.1, (4.10) and (4.18), one obtains that for any k, $i = 1,2$, and $\varepsilon > 0$ arbitrarily small,

$$\begin{aligned}
|\varphi_{p,i}^{(k)}(\sigma)-\psi_{p,i}^{(k)}(\sigma)| &\leq C(\varepsilon)\|\varphi_{p,i}-\psi_{p,i}\|_{H^{k+\frac{1}{2}+\varepsilon}(\Omega_i)} \\
&\leq C(\varepsilon)p^{2k+1+2\varepsilon}\|\varphi_{p,i}-\psi_{p,i}\|_{L_2(\Omega_i)} \\
&\leq C(\varepsilon)p^{2k+1+2\varepsilon}(\|u_i-\varphi_{p,i}\|_{L_2(\Omega_i)} \\
&\qquad +\|u_i-\psi_{p,i}\|_{L_2(\Omega_i)}) \\
&\leq C(\varepsilon)p^{-s+2k+1+3\varepsilon}.
\end{aligned} \tag{4.19}$$

Hence, (4.17), (4.19), and the fact that $\varphi_{p,1}^{(k)}(\sigma) = \varphi_{p,2}^{(k)}(\sigma)$, $0 \leq k \leq \ell$, imply that, for $0 \leq k \leq \min(\ell,\ell^*)$,

$$\begin{aligned}
|u_1^{(k)}(\sigma)-u_2^{(k)}(\sigma)| &\leq \sum_{i=1}^{2}|u_i^{(k)}(\sigma)-\varphi_{p,i}^{(k)}(\sigma)| \\
&\leq \sum_{i=1}^{2}(|u_i^{(k)}(\sigma)-\psi_{p,i}^{(k)}(\sigma)|+|\varphi_{p,i}^{(k)}(\sigma)-\psi_{p,i}^{(k)}(\sigma)|) \\
&\leq C(\varepsilon)p^{-s+2k+1+3\varepsilon} \to 0 \quad \text{as} \quad p \to \infty
\end{aligned}$$

provided that $\varepsilon > 0$ has been chosen small enough that $-s + 2k + 1 + 3\varepsilon < 0$. Thus $u_1^{(k)}(\sigma) = u_2^{(k)}(\sigma)$ for $0 \leq k \leq \min(\ell,\ell^*)$, which completes the proof.

5. Approximation by conforming piecewise polynomials continued: the case $n = 2$

As in the previous section, a number of technical lemmas precede the proofs of Theorems 3.1 and 3.2 for the case $n = 2$.

Lemma 5.1. Let s be a non-negative real number. If S is any triangle, then for each $\varphi_p \in P_p(S)$,

$$\|\varphi_p\|_{H^s(S)} \leq C(s)p^{2s}\|\varphi_p\|_{L_2(S)} \tag{5.1}$$

where C is independent of p and φ_p.

Pf: For each $x_1 \in I$, it follows from (4.1) that

$$\|\varphi_p(x_1,\cdot)\|^2_{H^s(I)} \leq Cp^{4s}\|\varphi_p(x_1,\cdot)\|^2_{L_2(I)}. \tag{5.2}$$

Similarly, for $x_2 \in I$,

$$\|\varphi_p(\cdot,x_2)\|^2_{H^s(I)} \leq Cp^{4s}\|\varphi_p(\cdot,x_2)\|^2_{L_2(I)}. \tag{5.3}$$

Integrating (5.2) and (5.3) with respect to x_1 and x_2, respectively, one obtains that for $Q = I^2$,

$$\|\varphi_p\|_{H^s(Q)} \leq Cp^{s}\|\varphi_p\|_{L_2(Q)}. \tag{5.4}$$

Using an affine mapping, one further obtains that (5.4) holds for any parallelogram Q. Since $S = \bigcup_{k=1}^{K} Q_k$ for some collection of parallelograms Q_k, (5.4) implies that

$$\|\varphi_p\|_{H^s(S)} \le \sum_{k=1}^{K} \|\varphi_p\|_{H^s(Q_k)} \le Cp^{2s} \sum_{k=1}^{K} \|\varphi_p\|_{L_2(Q_k)} \le Cp^{2s}\|\varphi_p\|_{L_2(S)}$$

which completes the proof.

Lemma 5.2. Let S be a triangle with sides $\gamma_1, \gamma_2, \gamma_3$. Let ℓ and p be non-negative integers, and for $k_1 = 0,\dots,\ell$ let $z_{p,k_1} \in P_p(\gamma_1)$ be such that $z_{p,k_1}^{(k_2)}$ vanishes at the endpoints of γ_1 for $0 \le k_2 \le \ell$. Let n and τ be unit vectors normal and tangent to γ_1, respectively. Then there exists $\psi_{2p} \in P_{2p}(S)$ such that, for $0 \le |\underline{k}| \le \ell$,

$$(5.5)\qquad \begin{aligned} \frac{\partial^{|\underline{k}|}}{\partial n^{k_1}\partial\tau^{k_2}}\psi_{2p} &= z_{p,k_1}^{(k_2)} \quad \text{on } \gamma_1 \\ \psi_{2p}^{(\underline{k})} &= 0 \qquad \text{on } \gamma_2 \text{ and } \gamma_3 \end{aligned}$$

and

$$(5.6)\qquad \|\psi_{2p}\|_{L_2(S)} \le C_1 p^{-(2\ell+1)}\|z_{p,\ell}\|_{L_2(\gamma_1)} + C_2(\varepsilon)\sum_{k=0}^{\ell-1} p^{-(2k+1)+\varepsilon}\|z_{p,k}\|_{L_2(\gamma_1)}$$

where C_1 and C_2 are independent of p. If $\ell = 0$ the second term in the right-hand side of (5.6) is omitted.

Pf: Without loss of generality, it may be assumed that

$$S = \{x : -1\le x_1\le 1,\ \alpha(x_1-1)-1\le x_2\le\beta(x_1-1)+1\}$$

for some numbers α, β and that

$$\gamma_1 = \{x : x_1 = 1, -1 \le x_2 \le 1\}.$$

By Lemma 4.3, for each $k = 0,\dots,\ell$ and each integer $p \ge \ell + 1$, there exists $\varphi_{p,k} \in P_p(I)$ satisfying (4.11) (with $\bar{p} = p$). Let

$$\psi_{2p,0}(x) = \varphi_{p,0}(x_1) z_{p,0}(x_2) \left(\frac{-\alpha(x_1-1)+x_2+1}{x_2+1} \; \frac{\beta(x_1-1)-x_2+1}{-x_2+1} \right)^{\ell+1}$$

and recursively define

$$\psi_{2p,k}(x) =$$

$$(z_{p,k}(x_2) - \psi^{(k,0)}_{2p,k-1}(1,x_2))\varphi_{p,k}(x_1) \left(\frac{-\alpha(x_1-1)+x_2+1}{x_2+1} \; \frac{\beta(x_1-1)-x_2+1}{-x_2+1} \right)^{\ell+1}$$

$$+ \psi_{2p,k-1}(x).$$

Setting $\psi_{2p} = \psi_{2p,\ell}$, one checks that ψ_{2p} satisfies (5.5), and if $\ell = 0$, (4.11) and the fact that $\left|\frac{-\alpha(x_1-1)+x_2+1}{x_2+1}\right|$ and $\left|\frac{\beta(x_1-1)-x_2+1}{-x_2+1}\right|$ are bounded on S imply that

$$\|\psi_{2p}\|_{L_2(S)} \le C p^{-1} \|z_{p,0}\|_{L_2(\gamma_1)}$$

which establishes (5.6) for $\ell = 0$. If $\ell > 0$, then by a well-known embedding result [10] together with Lemma 4.1, it follows that

$$\|\psi^{(k,0)}_{2p,k-1}(1,\cdot)\|_{L_2(\gamma_1)} \le C(\varepsilon)\|\psi_{2p,k-1}\|_{H^{k+\frac{1}{2}+\varepsilon}(S)}$$

$$\le C(\varepsilon)p^{2k+1+2\varepsilon}\|\psi_{2p,k-1}\|_{L_2(S)}.$$

Hence, for $k = 1,\ldots,\ell$,

$$\|\psi_{2p,k}\|_{L_2(S)} \le Cp^{-(2k+1)}\|\Xi_{p,k}\|_{L_2(\gamma_1)} + C(\varepsilon)p^{2\varepsilon}\|\psi_{2p,k-1}\|_{L_2(S)}$$

from which (5.6) follows.

Now let

$$S^* = \{x = (x_1,x_2) : -1<x_1<1,\ -x_1<x_2<1\}$$

and let q_ν, $1 \le \nu \le 3$, and γ_ν, $1 \le \nu \le 3$, denote the vertices and sides of S^*, respectively.

Lemma 5.3. Let ℓ be a non-negative integer and let $\alpha_{\underline{k},\nu}, 0 \le k_1,k_2 \le \ell$, $\nu = 1,2,3$, be real numbers. Then for any integer $p \ge 4(\ell+1)$, there exists $\psi_p \in P_p(S^*)$ such that

$$(5.7)\quad \psi_p^{(\underline{k})}(q_\nu) = \alpha_{\underline{k},\nu}, \qquad 0 \le k_1,k_2 \le \ell, \quad 1 \le \nu \le 3,$$

and for arbitrarily small $\varepsilon > 0$,

$$(5.8)\quad \|\psi_p\|_{L_2(S^*)} \le C(\varepsilon)\sum_{\nu=1}^{3}\sum_{k_1,k_2=0}^{\ell}|\alpha_{\underline{k},\nu}|p^{-2|\underline{k}|-2+\varepsilon},$$

$$(5.9)\quad \|\psi_p^{(\underline{m})}\|_{L_2(\gamma_\nu)} \le C(\varepsilon)\sum_{\nu=1}^{3}\sum_{k_1,k_2=0}^{\ell}|\alpha_{\underline{k},\nu}|p^{-2|\underline{k}|+2|\underline{m}|-1+\varepsilon},$$

$$0 \le |\underline{m}| \le \ell, \quad 1 \le \nu \le 3,$$

where C is independent of p. If $\ell = 0$, then (5.8) and (5.9) hold with $\varepsilon = 0$.

Pf: It suffices to prove the result for $\alpha_{\underline{k},\nu} = 0$, $0 \le k_1, k_2 \le \ell$, $\nu = 2,3$, since the general result may then be obtained via a superposition argument. Moreover, it may also be assumed that $d_1 = (1,1)$.

Fix $\underline{k} = (k_1, k_2)$ so that $0 \le k_1, k_2 \le \ell$. For $i = 1,2$, it follows from Lemma 4.4 that for each integer $\overline{p} \ge 2\ell + 2$ there exists $\psi_{\overline{p},k_i} \in P_p(I)$ such that

$$\psi^{(m)}_{\overline{p},k_i}(1) = \begin{cases} 1 & \text{if } m = k_i \\ 0 & \text{if } 0 \le m \le \ell,\ m \ne k_i, \end{cases}$$

$$\psi^{(m)}_{\overline{p},k_i}(-1) = 0, \qquad 0 \le m \le \ell,$$

and, for arbitrarily small $\varepsilon > 0$ (or $\varepsilon = 0$ if $k_i = \ell$),

$$\|\psi_{\overline{p},k_i}\|_{L_2(I)} \le C(\varepsilon) p^{-(2k_i+1)+\varepsilon}. \tag{5.10}$$

Setting $p = 2\overline{p}$ and

$$\psi_p(x) = \sum_{k_1,k_2=0}^{\ell} \alpha_{\underline{k},1} \psi_{\overline{p},k_1}(x_1)\psi_{\overline{p},k_2}(x_2), \qquad x = (x_1,x_2) \in S^*,$$

one obtains that ψ_p satisfies (5.7) and (5.8). To prove (5.9), one begins by observing that, by (5.10) and Lemma 4.1, for any real $s \ge 0$ and $\varepsilon > 0$,

$$(5.11)\quad \|\psi_{\bar{p},k_i}\|_{H^s(I)} \le Cp^{2s}\|\psi_{\bar{p},k_i}\|_{L_2(I)} \le C(\varepsilon)p^{-2k_i+2s-1+\varepsilon}.$$

Hence, for $0 \le |\underline{m}| \le \ell$,

$$\|\psi_p^{(\underline{m})}(1,\cdot)\|_{L_2(I)} \le \sum_{k_2=0}^{\ell} |\alpha_{(m_1,k_2),1}| \|\psi_{\bar{p},k_2}\|_{H^{m_2}(I)}$$

$$\le C(\varepsilon) \sum_{k_2=0}^{\ell} |\alpha_{(m_1,k_2),1}| p^{-2k_2+2m_2-1+\varepsilon}.$$

It is similarly shown that, for $0 \le |\underline{m}| \le \ell$,

$$\|\psi_p^{(\underline{m})}(\cdot,1)\|_{L_2(I)} \le C(\varepsilon) \sum_{k_1=0}^{\ell} |\alpha_{(k_1,m_2),1}| p^{-2k_1+2m_1-1+\varepsilon}.$$

Applying the Sobolev Lemma together with Lemma 4.1 and (5.11), it follows that for $x_1 \in I$, $\varepsilon > 0$, and $0 \le |\underline{m}| \le \ell$,

$$|\psi_p^{(\underline{m})}(x_1,-x_1)| \le C \sum_{k_1,k_2=0}^{\ell} |\alpha_{\underline{k},1}| |\psi_{\bar{p},k_1}^{(m_1)}(x_1)| \|\psi_{\bar{p},k_2}\|_{H^{m_2+\frac{1}{2}+\varepsilon}(I)}$$

$$\le C \sum_{k_1,k_2=0}^{\ell} |\alpha_{\underline{k},1}| p^{-2k_2+2m_2+2\varepsilon} |\psi_{\bar{p},k_1}^{(m_1)}(x_1)|$$

which then yields that

$$\int_{-1}^{1} |\psi_p^{(\underline{m})}(x_1,-x_1)|^2 dx_1 \le C \sum_{k_1,k_2=0}^{\ell} \alpha_{\underline{k},1}^2 p^{-4k_2+4m_2+4\varepsilon} \|\psi_{\bar{p},k_1}\|_{H^{m_1}(I)}^2$$

$$\le C \sum_{k_1,k_2=0}^{\ell} \alpha_{\underline{k},1}^2 \, p^{-4|\underline{k}|+4|\underline{m}|-2+6\varepsilon}$$

$$\le C\Big(\sum_{k_1,k_2=0}^{\ell} |\alpha_{\underline{k},1}| p^{-2|\underline{k}|+2|\underline{m}|-1+3\varepsilon}\Big)^2.$$

This completes the proof.

Lemma 5.4. Let s and s' be real numbers such that $s > 2s' > 0$ and $s, 2s' \neq \frac{1}{2}$ + an integer. Let ℓ be a non-negative integer. If $u \in Z^s(I^2)$, then for each integer $p \geq 4(\ell+1)$ there exists $\varphi_p \in P_p(S^*)$ such that for $\nu = 1,2,3$,

$$(5.12)\qquad \varphi_p^{(\underline{k})}(q_\nu) = \begin{cases} u^{(\underline{k})}(q_\nu) & \text{if } 0 \leq |\underline{k}| < \frac{s}{2} - 1 \\ 0 & \text{if } |\underline{k}| > \frac{s}{2} - 1,\ 0 \leq k_1, k_2 \leq \ell, \end{cases}$$

and for arbitrarily small $\varepsilon > 0$,

$$(5.13)\qquad \|u-\varphi_p\|_{H^{s'}(S^*)} \leq C(\varepsilon)p^{-s+2s'+\varepsilon}\|u\|_{Z^s(I^2)}.$$

Moreover, if $s \geq 1$, then for $0 \leq |\underline{m}| \leq \frac{s-1}{2}$ and $\nu = 1,2,3$,

$$(5.14)\qquad \|u^{(\underline{m})}-\varphi_p^{(\underline{m})}\|_{L_2(\gamma_\nu)} \leq C(\varepsilon)p^{-s+2|\underline{m}|+1+\varepsilon}\|u\|_{Z^s(I^2)}.$$

The constants C in (5.13) and (5.14) are independent of u and p.

Pf: Expand u in the series $\sum_{|\underline{m}|=0}^{\infty} a_{\underline{m}}\Phi_{\underline{m}}$ and for each non-negative integer p set $\xi_p = \sum_{|\underline{m}|=0}^{p} a_{\underline{m}}\Phi_{\underline{m}}$. By Theorem 2.4, one has that

$$(5.15)\qquad \|u-\xi_p\|_{Z^{2s'}(I^2)} \leq Cp^{-s+2s'}\|u\|_{Z^s(I^2)}.$$

By Lemma 3.1 and (5.15), it follows that for $\nu = 1,2,3$ and arbitrarily small $\varepsilon > 0$, if $0 \le |\underline{k}| < \frac{s}{2} - 1$, then

$$\begin{aligned} |u^{(\underline{k})}(q_\nu) - \xi_p^{(\underline{k})}(q_\nu)| &\le C(\varepsilon)\|u-\xi_p\|_{Z^{2|\underline{k}|+2+2\varepsilon}(I^2)} \\ &\le C(\varepsilon)p^{-s+2|\underline{k}|+2+2\varepsilon}\|u\|_{Z^s(I^2)}, \end{aligned} \tag{5.16}$$

and if $|\underline{k}| \ge \frac{s}{2} - 1$, then

$$\begin{aligned} |\xi_p^{(\underline{k})}(q_\nu)| &\le C(\varepsilon)\|\xi_p\|_{Z^{2|\underline{k}|+2+2\varepsilon}(I^2)} \\ &\le C(\varepsilon)\Big(\sum_{|\underline{m}|=0}^{p} a_{\underline{m}}^2|\underline{m}|^{2s}|\underline{m}|^{-2s+4|\underline{k}|+4+4\varepsilon}\Big)^{1/2} \\ &\le C(\varepsilon)p^{-s+2|\underline{k}|+2+2\varepsilon}\|u\|_{Z^s(I^2)}. \end{aligned} \tag{5.17}$$

Furthermore, a well-known embedding result [10] together with Lemma 2.6 and (5.15) implies that if $0 \le |\underline{m}| \le \frac{s-1}{2}$, then for $\nu = 1,2,3$ and arbitrarily small $\varepsilon > 0$,

$$\begin{aligned} \|u^{(\underline{m})} - \xi_p^{(\underline{m})}\|_{L_2(\gamma_\nu)} &\le C(\varepsilon)\|u-\xi_p\|_{H^{|\underline{m}|+\frac{1}{2}+\varepsilon}(I^2)} \\ &\le C(\varepsilon)\|u-\xi_p\|_{Z^{2|\underline{m}|+1+2\varepsilon}(I^2)} \\ &\le C(\varepsilon)p^{-s+2|\underline{m}|+1+2\varepsilon}\|u\|_{Z^s(I^2)}. \end{aligned} \tag{5.18}$$

By Lemma 5.3, for each integer $p \ge 4(\ell+1)$ there exists $\psi_p \in \mathcal{P}_p(I^2)$ such that, for $\nu = 1,2,3$,

$$(5.19)\quad \psi_p^{(\underline{k})}(q_\nu) = \begin{cases} u^{(\underline{k})}(q_\nu) - \xi_p^{(\underline{k})}(q_\nu) & \text{if } 0 \le |\underline{k}| < \frac{s}{2} - 1 \\ -\xi_p^{(\underline{k})}(q_\nu) & \text{if } |\underline{k}| \ge \frac{s}{2} - 1,\ 0 \le k_1, k_2 \le \ell, \end{cases}$$

and for arbitrarily small $\varepsilon > 0$,

$$(5.20)\quad \|\psi_p\|_{L_2(S^*)} \le C(\varepsilon) \sum_{\nu=1}^{3} \sum_{k_1,k_2=0}^{\ell} |\psi_p^{(\underline{k})}(q_\nu)|\, p^{-2|\underline{k}|-2+\varepsilon},$$

$$(5.21)\quad \|\psi_p^{(\underline{m})}\|_{L_2(\gamma_\nu)} \le C(\varepsilon) \sum_{\nu'=1}^{3} \sum_{k_1,k_2=0}^{\ell} |\psi_p^{(\underline{k})}(q_{\nu'})|\, p^{-2|\underline{k}|+2|\underline{m}|-1+\varepsilon},$$
$$0 \le |\underline{m}| \le \ell,\ 1 \le \nu \le 3.$$

Set $\varphi_p = \xi_p + \psi_p$. Then φ_p satisfies (5.12),and (5.14) follows from (5.18), (5.19), (5.21). By Lemmas 2.6 and 5.1 together with (5.15)-(5.17), (5.19) and (5.20), it follows that

$$\begin{aligned} \|u-\varphi_p\|_{H^{s'}(S^*)} &\le \|u-\xi_p\|_{H^{s'}(I^2)} + \|\psi_p\|_{H^{s'}(S^*)} \\ &\le C(\|u-\xi_p\|_{Z^{2s'}(I^2)} + p^{2s'}\|\psi_p\|_{L_2(S^*)}) \\ &\le Cp^{-s+2s'+\varepsilon}\|u\|_{Z^s(I^2)} \end{aligned}$$

which proves (5.13).

<u>Pf. of Theorem 3.1.</u> It follows from Lemma 5.4 that for each integer $p \ge 4(\ell+1)$ and each $i = 1,\ldots,M$, there exists $\varphi_{p,i} \in P_p(\Omega_i)$ such that, for $j = 1,2,3,$

$$(5.22) \quad \varphi_{p,i}^{(\underline{m})}(\sigma_{i,j}) = \begin{cases} u^{(\underline{m})}|_{\Omega_i}(\sigma_{i,j}) & \text{if } |\underline{m}| \le \ell^* \\ 0 & \text{if } |\underline{m}| > \ell^* \end{cases}$$

and for arbitrarily small $\varepsilon > 0$,

$$(5.23) \quad \|u-\varphi_{p,i}\|_{H^{s'}(\Omega_i)} \le C(\varepsilon)p^{-s+2s'+\varepsilon}\|u\|_{Z^s(\Omega;\Delta)}$$

$$(5.24) \quad \|u^{(\underline{m})}-\varphi_{p,i}^{(\underline{m})}\|_{L_2(\gamma_{i,j})} \le C(\varepsilon)p^{-s+2|\underline{m}|+1+\varepsilon}\|u\|_{Z^s(\Omega;\Delta)},$$

$$0 \le |\underline{m}| \le \frac{s-1}{2}.$$

Suppose that Ω_1 and Ω_2 are adjacent triangles of Δ, and let γ denote their common side. Let n and τ denote unit vectors normal and tangent to γ, respectively. For $0 \le m_1 \le \ell$, set

$$(5.25) \quad z_{p,m_1} = \frac{\partial^{m_1}}{\partial n^{m_1}} \varphi_{p,1}|_\gamma - \frac{\partial^{m_1}}{\partial n^{m_1}} \varphi_{p,2}|_\gamma .$$

Since $u \in C^\ell(\bar{\Omega})$ and $u|_{\Omega_i} \in C^{\ell^*}(\bar{\Omega}_i)$, $i = 1,2$, it follows from (5.22) that, for $0 \le m_1,m_2 \le \ell$, $z_{p,m_1}^{(m_2)}$ vanishes at the endpoints of γ. By Lemma 5.2 there exists $\psi_{2p} \in P_{2p}(\Omega_1)$ such that for $0 \le |\underline{m}| \le \ell$,

$$(5.26) \quad \begin{aligned} \frac{\partial^{|\underline{m}|}}{\partial n^{m_1}\partial\tau^{m_2}} \psi_{2p} &= z_{p,m_1}^{(m_2)} && \text{on } \gamma, \\ \psi_{2p}^{(\underline{m})} &= 0 && \text{on } \partial\Omega_1\backslash\gamma, \end{aligned}$$

and

$$\|\psi_{2p}\|_{L_2(\Omega_1)} \leq C_1 p^{-(2\ell+1)} \|z_{p,\ell}\|_{L_2(\gamma)} + C_2(\varepsilon) \sum_{m_1=0}^{\ell-1} p^{-(2m_1+1)+\varepsilon} \cdot \|z_{p,m_1}\|_{L_2(\gamma)}.$$

From (5.24) and the fact that $u \in C^{\ell}(\bar{\Omega})$, one obtains that for $m_1 = 0,\ldots,\ell$,

$$\|z_{p,m_1}\|_{L_2(\gamma)} \leq \left\|\frac{\partial^{m_1}}{\partial n^{m_1}}(u-\varphi_{p,1})\right\|_{L_2(\gamma)} + \left\|\frac{\partial^{m_1}}{\partial n^{m_1}}(u-\varphi_{p,2})\right\|_{L_2(\gamma)}$$

$$\leq C(\varepsilon) p^{-s+2m_1+1+\varepsilon} \|u\|_{Z^s(\Omega;\Delta)}.$$

Hence,

$$(5.27) \qquad \|\psi_{2p}\|_{L_2(\Omega_1)} \leq C(\varepsilon) p^{-s+\varepsilon} \|u\|_{Z^s(\Omega,\Delta)}.$$

Replacing $\varphi_{p,1}$ on Ω_1 by $\varphi_{2p,1} = \varphi_{p,1} - \psi_{2p}$, it follows from (5.25) and (5.26) that $\varphi^{(\underline{m})}_{2p,1} = \varphi^{(\underline{m})}_{p,2}$, $0 \leq |\underline{m}| \leq \ell$, on γ and that $\varphi^{(\underline{m})}_{2p,1} = \varphi^{(\underline{m})}_{p,1}$, $0 \leq |\underline{m}| \leq \ell$, on $\partial\Omega_1\backslash\gamma$.
Moreover, (5.23) and (5.27) together with Lemma 5.1 yield that

$$\|u-\varphi_{2p,1}\|_{H^{s'}(\Omega_1)} \leq \|u-\varphi_{p,1}\|_{H^{s'}(\Omega_1)} + \|\psi_{2p}\|_{H^{s'}(\Omega_1)}$$

$$\leq \|u-\varphi_{p,1}\|_{H^{s'}(\Omega_1)} + Cp^{2s'}\|\psi_{2p}\|_{L_2(\Omega_1)}$$

$$\leq C(\varepsilon)p^{-s+2s'+\varepsilon}\|u\|_{Z^s(\Omega;\Delta)}.$$

The proof is completed by repeating the above procedure for all remaining pairs of adjacent triangles in Δ.

Pf. of Theorem 3.2. The proof is analogous to that for the case $n = 1$. By Theorem 2.5, it holds that $u \in Z^{s-\varepsilon}(\Omega;\Delta) = Z^{s-\varepsilon}_{-1}(\Omega;\Delta)$ for arbitrarily small $\varepsilon > 0$, so it only remains to establish the regularity of u across the common side γ of two adjacent triangles of Δ, say Ω_1 and Ω_2.

Let $u_i = u|_{\Omega_i}$, $i = 1,2$. Since $u \in Z^{s-\varepsilon}(\Omega;\Delta)$, it follows from Lemma 5.4 that for each non-negative integer p there exists $\psi_{p,i} \in P_p(\Omega_i)$ such that for $\varepsilon > 0$ arbitrarily small,

$$(5.28) \qquad \|u_i-\psi_{p,i}\|_{L_2(\Omega_i)} \le C(\varepsilon)p^{-s+\varepsilon}\|u\|_{Z^{s-\varepsilon}(\Omega,\Delta)}, \qquad i = 1,2,$$

and for $0 \le |\underline{k}| \le \ell^*$,

$$(5.29) \qquad \|u_i^{(\underline{k})}-\psi_{p,i}^{(\underline{k})}\|_{L_2(\gamma)} \le C(\varepsilon)p^{-s+2|\underline{k}|+1+\varepsilon}\|u\|_{Z^{s-\varepsilon}(\Omega,\Delta)}, \quad i = 1,2.$$

Let $\varphi_{p,i} = \varphi_p|_{\Omega_i}$, $i = 1,2$. One obtains from (3.3), (5.28), and Lemma 5.1 that for $0 \le |\underline{k}| \le \ell^*$, $i = 1,2$, and $\varepsilon > 0$ arbitrarily small,

$$\begin{aligned}
\|\psi_{p,i}^{(\underline{k})}-\varphi_{p,i}^{(\underline{k})}\|_{L_2(\gamma)} &\le C(\varepsilon)\|\psi_{p,i}-\varphi_{p,i}\|_{H^{|\underline{k}|+\frac{1}{2}+\varepsilon}(\Omega_i)} \\
&\le C(\varepsilon)p^{2|\underline{k}|+1+2\varepsilon}\|\psi_{p,i}-\varphi_{p,i}\|_{L_2(\Omega_i)} \\
(5.30) \qquad &\le C(\varepsilon)p^{2|\underline{k}|+1+2\varepsilon}(\|u_i-\psi_{p,i}\|_{L_2(\Omega_i)} + \\
&\qquad \|u_i-\varphi_{p,i}\|_{L_2(\Omega_i)}) \\
&\le C(\varepsilon)p^{-s+2|\underline{k}|+1+3\varepsilon}.
\end{aligned}$$

Hence, by (5.29) and (5.30) together with the fact that $\phi_{p,i}^{(\underline{k})} = \varphi_{p,i}^{(\underline{k})}$ on γ for $0 \le |\underline{k}| \le \ell$, it follows that for $0 \le |\underline{k}| \le \min(\ell,\ell^*)$,

$$\|u_1^{(\underline{k})} - u_2^{(\underline{k})}\|_{L_2(\gamma)} \le \sum_{i=1}^{2} \|u_i^{(\underline{k})} - \varphi_{p,i}^{(\underline{k})}\|_{L_2(\gamma)}$$

$$\le \sum_{i=1}^{2} \left(\|u_i^{(\underline{k})} - \phi_{p,i}^{(\underline{k})}\|_{L_2(\gamma)} + \|\phi_{p,i}^{(\underline{k})} - \varphi_{p,i}^{(\underline{k})}\|_{L_2(\gamma)} \right)$$

$$\le C(\varepsilon) p^{-s+2|\underline{k}|+1+3\varepsilon} \to 0 \quad \text{as} \quad p \to \infty$$

provided that $\varepsilon > 0$ has been chosen small enough that $-s + 2|\underline{k}| + 1 + 3\varepsilon < 0$. Thus $u_1^{(\underline{k})} = u_2^{(\underline{k})}$ on γ for $0 \le |\underline{k}| \le \min(\ell,\ell^*)$ which completes the proof.

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